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JET PROPULSION LAB PASADENA CA M W RENNIE 22 JAN 87
JPL-D-4621 NAS7-918

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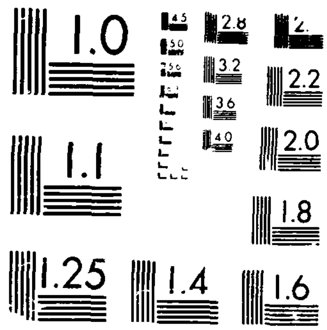

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U.S. ARMY INTELLIGENCE CENTER AND SCHOOL
SOFTWARE ANALYSIS AND MANAGEMENT SYSTEM

NORMALIZATION FACTORS USED IN ESTIMATING VARIANCE

TECHNICAL MEMORANDUM NO. 24

MARC

Mathematical Analysis Research Corporation

22 January 1987

National Aeronautics and
Space Administration

JPL

JET PROPULSION LABORATORY
California Institute of Technology
Pasadena, California

JPL D-4621
ALGO_PUB_0094

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
ALGO-PUB-0094

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
Technical Memo 24, "Normalization Factors Used in Estimating Variance"

5. TYPE OF REPORT & PERIOD COVERED
FINAL

6. PERFORMING ORG. REPORT NUMBER
D-4621

7. AUTHOR(s)
Mathematical Analysis Research Corp. (MARC)

8. CONTRACT OR GRANT NUMBER(s)
NAS7-918

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Jet Propulsion Laboratory, ATTN: 171-209
California Institute of Technology
4800 Oak Grove, Pasadena, CA 91109

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
RE 182 AMEND #187

11. CONTROLLING OFFICE NAME AND ADDRESS
Commander, USAICS
ATTN: ATSI-CD-SF
Ft. Huachuca, AZ 85613-7000

12. REPORT DATE
22 Jan 87

13. NUMBER OF PAGES
12

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
Commander, USAICS
ATTN: ATSI-CD-SF
Ft. Huachuca, AZ 85613-7000

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
NONE
None

16. DISTRIBUTION STATEMENT (of this Report)
Approved for Public Dissemination

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES
Prepared by Jet Propulsion Laboratory for the US Army Intelligence Center and School's Combat Developer's Support Facility.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Perpendicular, Weighted Perpendicular, Minimization of Squared Angular Error, Minimization of Sine of Squared Angular Error Algorithms, Variance, Normalization Factor

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
This report determines the correct factor (N, N-1, N-2, or others - where N is the number of lines-of-bearing) used in estimating variance from fixing data. Four methods of estimating fixes are considered: Perpendicular, Weighted Perpendicular, Minimization of Squared Angular Error, and Minimization of Sine of Squared Angular Error. A table of results from numerical experiments using the Perpendicular method is presented. The formula for first order

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

estimation of variance for the other three methods is derived. Evaluation of this formula and proof that the normalizing factor in all three cases is N-2 is relegated to an appendix.

7057-69

U.S. ARMY INTELLIGENCE CENTER AND SCHOOL
Software Analysis and Management System

Normalization Factors Used In Estimating Variance

Technical Memorandum No. 24

22 January 1987

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JPL D-4621

PREFACE

The work described in this publication was performed by the Mathematical Analysis Research Corporation (MARC) under contract to the Jet Propulsion Laboratory, an operating division of the California Institute of Technology. This activity is sponsored by the Jet Propulsion Laboratory under contract NAS7-918, RE182, A187 with the National Aeronautics and Space Administration, for the United States Army Intelligence Center and School.

This specific work was performed in accordance with the FY-87 statement of work (SOW #2).

Normalization Factors Used In Estimating Variance

INTRODUCTION

While investigating Quickfix, two questions arose concerning the part of the code where the estimate of variance is computed. (The estimate is only used if the estimated standard deviation derived from the variance is bigger than 3.) One question concerned the impact of truncation and shall not be discussed further here. The other question concerned the division by n-3 that occurs in the calculation where n is the number of lines of bearing being used. If n-3 is wrong it affects ellipse size estimates. Simple one variable problems use n-1, but since there are two coordinates to be determined one might expect to see n-2. On further reflection, the usual analysis does not directly apply to fixing so an investigation seemed necessary.

It turns out that use of the expected value of the estimator implies:

i) n-2 is the correct term to divide by for Minimization of Square Angular Error methods and sine variations such as FFIX. It is also the correct term for Weighted Perpendicular.

ii) n-2 only applies to the Perpendicular Method in the case where all sensors are approximately the same distance from the target. In other cases this estimator is unstable but probably conservative.

As an example, suppose one sensor is very close to the emitter in comparison with the other sensors. Then

a) the close sensor doesn't influence the point estimate much

b) however, with a close sensor any size measured (not real) angular error is possible. If sensors are close enough estimates could even violate common sense since close sensors are nearly ignored. The result is that division by n-2 is conservative and error ellipses would be larger than necessary at least with regard to this one issue.

Applicability of F statistics and tail behavior is more difficult to analyze and not done here. At the very least a closer look is merited. Analysis of the behavior of the variance estimator alone does not give a complete picture.

On the next page a few examples are given. The analysis for the different methods follows in the rest of the report.

I. EXAMPLES

All methods were computed but only Perpendicular is listed as other methods all yielded n-2 (where n is the number of points) as expected.

Put the Emitter on the y-axis at (0,20) and the sensors ...

SENSOR LOCATIONS	n-2	Perpendicular Method
on the x-axis at (-20,0), (0,0), and (20,0)	1	1.125
on the x-axis at (0,0), (20,0), and (40,0)	1	1.286
on the x-axis at (0,0), (40,0), and (80,0)	1	1.620
on the x-axis at (0,0), (20,0) and (400,0)	1	13.323
equidistant from the emitter at (0,0), (12,4), and (16,8)	1	1.000
on the x-axis at (-20,0),(-16,0),(-12,0), (-8,0),(-4,0),(0,0),(4,0), (8,0),(12,0),(16,0), and (20,0)	9	9.095
on the x-axis at (-40,0),(-32,0),(-24,0), (-16,0),(-8,0),(0,0),(8,0), (16,0),(24,0),(32,0), and (40,0)	9	9.492

This means for example that in the (0,0),(20,0), and (400,0) case one should divide the statistic for variance by 13.323 whereas in fact it is divided by 1. Hence the resulting error ellipse estimate would be too large from this point of view by a factor of 13.323 for area or the square root of 13.323 too larg in each direction.

II. EXPECTED VALUE OF THE UNNORMALIZED ANGULAR VARIANCE ESTIMATOR STATISTIC

Definitions

(X,Y) = true location of the emitter

$\theta_k(X,Y)$ = true bearing from the kth sensor to the emitter (X,Y)

$\hat{\theta}_k$ = observed bearing from the kth sensor (multiple readings are treated as coming from different sensors)

$(x,y) = (x(\hat{\theta}_1,\ldots,\hat{\theta}_n), y(\hat{\theta}_1,\ldots,\hat{\theta}_n))$ = estimated location of the emitter

$\theta_k(x,y)$ = bearing from the kth sensor to the estimate (x,y)

σ = standard deviation of the angular measurement of the LOBs in radians (multiply by $\pi/180$ if in degrees)

ε_k = error in kth LOB = $\hat{\theta}_k - \theta_k(X,Y)$ (assume independent ε_k)

(x_k,y_k) = sensor location of kth LOB

$r_k^2 = (x-x_k)^2+(y-y_k)^2 \qquad R_k^2=(X-x_k)^2+(Y-y_k)^2$

Calculations

The following approximation will be the basis for subsequent calculations

$$\theta_k(x,y)-\hat{\theta}_k \sim [\theta_k(X,Y)+\sum_{j=1}^{n}(\frac{\partial\theta_k(x,y)}{\partial x}\frac{\partial x}{\partial\varepsilon_j}+\frac{\partial\theta_k(x,y)}{\partial y}\frac{\partial y}{\partial\varepsilon_j})\varepsilon_j]-[\theta_k(X,Y)+\varepsilon_k]$$

where all of the partial derivatives indicated must be evaluated at the true point or zero error. (Note that this approximation is based upon a Taylor Series expansion.) To evaluate this expression one also needs to recall that

$$\theta_k(x,y) = \text{Arctan}((x-x_k)/(y-y_k))$$

and hence

$$\frac{\partial\theta_k(x,y)}{\partial x} = (y-y_k)/r_k^2$$

$$\frac{\partial\theta_k(x,y)}{\partial y} = -(x-x_k)/r_k^2$$

Substituting, evaluating at the true and simplifying one gets

$$\theta_k(x,y)-\hat{\theta}_k \sim [\sum_{j=1}^{n}((Y-y_k)\frac{\partial x}{\partial\varepsilon_j}-(X-x_k)\frac{\partial y}{\partial\varepsilon_j})\varepsilon_j]/R_k^2-[\varepsilon_k]$$

And hence

$$E(\sum_{k=1}^{n}(\theta_k(x,y)-\hat{\theta}_k)^2) \sim \sum_{k=1}^{n}[\sum_{j=1}^{n}\{((Y-y_k)\frac{\partial x}{\partial\varepsilon_j}-(X-x_k)\frac{\partial y}{\partial\varepsilon_j})\}^2\sigma^2/R_k^4]$$

$$-2\sum_{k=1}^{n}[((Y-y_k)\frac{\partial x}{\partial\varepsilon_k}-(X-x_k)\frac{\partial y}{\partial\varepsilon_k})]\sigma^2/R_k^2+n\sigma^2$$

III. LEAST SQUARES TYPE FIX METHODS

Part of the analyis of these methods is independent of which method is being analyzed.

Extra definitions that apply to Least Square based methods

$L_k = L_k(x,y,\theta_k)$=the 'squared error term' corresponding to the kth LOB and a location estimate of (x,y) for the least square method.

$L(x,y,\theta_1,\dots,\theta_n) = \sum_{i=1}^{n} L_k$=the sum of squares that the method minimizes

Let $b_k=\partial L_k/\partial x$ $c_k=\partial L_k/\partial y$ $d_k=\partial^2 L_k/\partial x^2$ $e_k=\partial^2 L_k/\partial x\partial y$ $f_k=\partial^2 L_k/\partial y^2$

where each term above is evaluated at the true angles and point.

Let $'$ notation represent derivatives with respect to θ_k for b_k and c_k.
Let $Q=1/(\Sigma d_i \Sigma f_i - \Sigma e_i \Sigma e_i)$.

These include the Perpendicular method, minimization of angular error, and the sine variation of minimization of angular error used by FFIX

Assume that x,y are defined implicitly as the minimum of

$$L=\Sigma\, L_k(x,y,\theta_k)$$

in the sense that $\partial L/\partial x=\partial L/\partial y=0$

One can show (as MARC has in it's report "Two Dimensional Uncorrelated Bias in Fix Algorithms") that the partial derivatives evaluated at the true are

$$\text{so,}\quad \begin{bmatrix} \partial x/\partial \varepsilon_i \\ \partial y/\partial \varepsilon_i \end{bmatrix} = \frac{-1}{(\Sigma d_k)(\Sigma f_k)-(\Sigma e_k)^2} \begin{bmatrix} \Sigma f_k & -\Sigma e_k \\ -\Sigma e_k & \Sigma d_k \end{bmatrix} \begin{bmatrix} b_i' \\ c_i' \end{bmatrix}$$

The value of the first partials of x and y for different Least Square based methods and derivations of the correct value term to divide by are listed in the the first portion of the Appendix.

Appendix

Minimization of Square Angular Error Method

$L_k = [\text{Arctan}((x-x_k)/(y-y_k))-\theta_k]^2$

$b'_k = -2(Y-y_k)/R_k^2$

$c'_k = 2(X-x_k)/R_k^2$

$d_k = 2(Y-y_k)^2/R_k^4$

$e_k = -2(X-x_k)(Y-y_k)/R_k^4$

$f_k = 2(X-x_k)^2/R_k^4$

Perpendicular Method

$L_k = [(x-x_k)^2\cos^2\theta_k+(y-y_k)^2\sin^2\theta_k-2(y-y_k)(x-x_k)\sin\theta_k\cos\theta_k]$

$b'_k = -2(Y-y_k)$

$c'_k = 2(X-x_k)$

$d_k = 2(Y-y_k)^2/R_k^2$

$e_k = -2(X-x_k)(Y-y_k)/R_k^2$

$f_k = 2(X-x_k)^2/R_k^2$

Comparison with similar terms for the Minimization of Square Angular Error method (above) shows that the only difference is a factor of R_k^2. Examination of the formula for partial derivatives of x and y with respect to measurement error evaluated at the true shows that if all R_k are the same then the difference between the perpendicular method and minimization of cancels out and these evaluated partial derivatives would be the same. As a result the n-2 normalization factor that is derived for Minimization of Angular Error also applies to the Perpendicular Method when the distance of all sensors from the emitter are equal.

Sine of Error Minimization Method

The partial derivatives for this method are identical to the ones just shown for the Minimization of Angular Error method. Thus, first-order terms are identical for both methods. However, the a_k is as follows.

$L_k = [(x-x_k)^2\cos^2\theta_k+(y-y_k)^2\sin^2\theta_k-2(y-y_k)(x-x_k)\sin\theta_k\cos\theta_k]/[(x-x_k)^2+(y-y_k)^2]$

Demonstration that n-2 applies to Minimization of Square Angular Error

For this method, $(b_i')^2=2d_i$, $(c_i')^2=2f_i$, and $b_i'c_i'=2e_i$.

Thus

$$Q^2 \sum_{j=1}^{n}\{((Y-y_k)\frac{\partial x}{\partial \varepsilon_j}-(X-x_k)\frac{\partial y}{\partial \varepsilon_j})\}^2 =$$

$$= \sum_{j=1}^{n}\ [\{(Y-y_k)\Sigma f_i+(X-x_k)\Sigma e_i\}b_j'-\{(Y-y_k)\Sigma e_i+(X-x_k)\Sigma d_i\}c_j']^2$$

$$= \{\{(Y-y_k)\Sigma f_i+(X-x_k)\Sigma e_i\}^2 \sum_{j=1}(b_j')^2$$
$$-2\{(Y-y_k)\Sigma f_i+(X-x_k)\Sigma e_i\}\{(Y-y_k)\Sigma e_i+(X-x_k)\Sigma d_i\}\sum_{j=1}^{n}b_j'c_j'$$
$$+\{(Y-y_k)\Sigma e_i+(X-x_k)\Sigma d_i\}^2 \sum_{j=1}^{n}(c_j')^2$$

$$= \{(Y-y_k)\Sigma f_i+(X-x_k)\Sigma e_i\}^2 \sum_{j=1}2d_j$$
$$-2\{(Y-y_k)\Sigma f_i+(X-x_k)\Sigma e_i\}\{(Y-y_k)\Sigma e_i+(X-x_k)\Sigma d_i\}\sum_{j=1}^{n}2e_j$$
$$+\{(Y-y_k)\Sigma e_i+(X-x_k)\Sigma d_i\}^2 \sum_{j=1}^{n}2f_j$$

$$= [\{(Y-y_k)\Sigma f_i\}^2+2(Y-y_k)(X-x_k)\Sigma e_i\Sigma f_i\}+\{(X-x_k)\Sigma e_i\}^2]\sum_{j=1}^{n}2d_j$$
$$-2[(Y-y_k)^2\Sigma f_i\Sigma e_i+(X-x_k)(Y-y_k)\{(\Sigma e_i)^2+\Sigma d_i\Sigma f_i\}+(X-x_k)^2\Sigma d_i\Sigma e_i]\sum_{j=1}^{n}2e_j$$
$$+[\{(Y-y_k)\Sigma e_i\}^2+2(Y-y_k)(X-x_k)\Sigma d_i\Sigma e_i\}+\{(X-x_k)\Sigma d_i\}^2]\sum_{j=1}^{n}2f_j$$

$$= (Y-y_k)^2\{2\Sigma f_i\Sigma f_i\Sigma d_i-2\Sigma f_i\Sigma e_i\Sigma e_i\}$$
$$+(X-x_k)(Y-y_k)\{4\Sigma d_i\Sigma e_i\Sigma f_i-4\Sigma e_i\Sigma e_i\Sigma e_i\}$$
$$+(X-x_k)^2\{-2\Sigma e_k\Sigma e_k\Sigma d_k+2\Sigma d_k\Sigma d_k\Sigma f_k\}$$

However since Q factors out of the last expression one has

$$\sum_{j=1}^{n}\{((Y-y_k)\frac{\partial x}{\partial \varepsilon_j}-(X-x_k)\frac{\partial y}{\partial \varepsilon_j})\}^2= 2[(Y-y_k)^2\Sigma f_i+2(X-x_k)(Y-y_k)\Sigma e_i+(X-x_k)^2\Sigma d_i]/Q$$

Hence

$$\sum_{k=1}^{n}[\sum_{j=1}^{n}\{((Y-y_k)\frac{\partial x}{\partial \varepsilon_j}-(X-x_k)\frac{\partial y}{\partial \varepsilon_j})\}^2\sigma^2/R_k^4] = [\ \Sigma_k d_k\ \Sigma_i f_i-2\Sigma e_k\Sigma e_i+\Sigma_k f_k\Sigma_i d_i]\sigma^2/Q$$
$$= 2Q\sigma^2/Q$$
$$= 2\sigma^2$$

The other term

$$\sum_{k=1}^{n}\{((Y-y_k)\frac{\partial x}{\partial \varepsilon_j}-(X-x_k)\frac{\partial y}{\partial \varepsilon_j})\}\sigma^2/R_k^2$$
$$=\sum_{k=1}^{n}[\{(Y-y_k)\Sigma f_i+(X-x_k)\Sigma e_i\}b_k'-\{(Y-y_k)\Sigma e_i+(X-x_k)\Sigma d_i\}c_k']\sigma^2/R_k^2$$
$$= -[-\Sigma d_k\Sigma f_k+\Sigma e_k\Sigma e_k+\Sigma e_k\Sigma e_k-\Sigma f_k\Sigma d_k]\sigma^2/Q$$
$$= 2\sigma^2$$

$$E(\sum_{k=1}^{n}(\theta_k(x,y)-\hat{\theta}_k)^2) \sim 2\sigma^2-2(2\sigma^2)+n\sigma^2=(n-2)\sigma^2$$

<u>Weighted Perpendicular</u> (Not Definable in Terms of Least Squares)

The Weighted Perpendicular Method is not definable as a least square's method and so a separate formula needs to be derived for first-order bias. It is necessary to establish notation for this case.

<u>Definitions</u>

$$a_i = \cos^2\hat{\theta}_i \quad A_i = \cos^2\theta_i(X,Y)=(Y-y_i)^2/R_i^2 \quad a = \Sigma a_i/r_i^2 \quad A = \Sigma A_i/R_i^2$$

$$b_i = \sin^2\hat{\theta}_i \quad B_i = \sin^2\theta_i(X,Y)=(X-x_i)^2/R_i^2 \quad b = \Sigma b_i/r_i^2 \quad B = \Sigma B_i/R_i^2$$

$$c_i = \sin\hat{\theta}_i\cos\hat{\theta}_i \quad C_i=(X-x_i)(Y-y_i)/R_i^2 \quad c = \Sigma c_i/r_i^2 \quad C = \Sigma C_i/R_i^2$$

$$d_i = a_i x_i - c_i y_i \quad D_i = A_i x_i - C_i y_i \quad d = \Sigma d_i/r_i^2 \quad D = \Sigma D_i/R_i^2$$

$$e_i = -c_i x_i + b_i y_i \quad E_i = -C_i x_i + B_i y_i \quad e = \Sigma e_i/r_i^2 \quad E = \Sigma E_i/R_i^2$$

$$s_i = \begin{bmatrix} a_i & -c_i \\ -c_i & b_i \end{bmatrix} \qquad \Sigma s_i/r_i^2 = \begin{bmatrix} a & -c \\ -c & b \end{bmatrix}$$

$$(x,y) = (\Sigma s_i/r_i^2)^{-1}(\Sigma s_i/r_i^2 \begin{bmatrix} x_i \\ y_i \end{bmatrix}) = \frac{1}{ab-c^2}\begin{bmatrix} b & c \\ c & a \end{bmatrix}\begin{bmatrix} d \\ e \end{bmatrix}$$

First Partials:

$$\partial r_i^2/\partial\hat{\theta}_k = 2(x-x_i)(\partial x/\partial\hat{\theta}_k) + 2(y-y_i)(\partial y/\partial\hat{\theta}_k)$$

$$\partial a/\partial\hat{\theta}_k = -2c_k/r_k^2 - \Sigma(a_i/r_i^4\cdot\partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial b/\partial\hat{\theta}_k = 2c_k/r_k^2 - \Sigma(b_i/r_i^4\cdot\partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial c/\partial\hat{\theta}_k = (a_k-b_k)/r_k^2 - \Sigma(c_i/r_i^4\cdot\partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial d/\partial\hat{\theta}_k = [-2c_k x_k + (b_k-a_k)y_k]/r_k^2 - \Sigma(d_i/r_i^4\cdot\partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial e/\partial\hat{\theta}_k = [(b_k-a_k)x_k + 2c_k y_k]/r_k^2 - \Sigma(d_i/r_i^4\cdot\partial r_i^2/\partial\hat{\theta}_k)$$

Differentiating the defining equation:

$$\begin{bmatrix} \partial x/\partial\hat{\theta}_k \\ \partial y/\partial\hat{\theta}_k \end{bmatrix} = \frac{-1}{(ab-c^2)^2}(a(\partial b/\partial\theta_k)+b(\partial a/\partial\theta_k)-2c(\partial c/\partial\theta_k))\begin{bmatrix} bd+ce \\ cd+ae \end{bmatrix}$$

$$+\frac{1}{(ab-c^2)}\begin{bmatrix} b(\partial d/\partial\theta_k) + d(\partial b/\partial\theta_k) + c(\partial e/\partial\theta_k) + e\partial c/\partial\theta_k) \\ c(\partial d/\partial\theta_k) + d(\partial c/\partial\theta_k) + a(\partial e/\partial\theta_k) + e(\partial a/\partial\theta_k) \end{bmatrix}$$

Simplifying the defining equation:

Let $f_k = 2[b^2d+bce-ace-c^2d]c_k + [2cbd+c^2e+abe](a_k-b_k)$
$+ [b^2a-c^2b](-2c_kx_k + (b_k-a_k)y_k) + [c(ab-c^2)]((b_k-a_k)x_k + 2c_ky_k)$

$h_k = [-b^2d-bce]a_k+[-ace-c^2d]b_k+[2cbd+c^2e+abe]c_k+[b^2a-c^2b]d_k+[c(ab-c^2)]e_k$

Let $g_k = 2[bcd+c^2e-acd-a^2e]c_k + [2ace+c^2d+abd](a_k-b_k)$
$+ [c(ab-c^2)](-2c_kx_k + (b_k-a_k)y_k) + [a(ab-c^2)]((b_k-a_k)x_k + 2c_ky_k)$

$i_k = [-bcd-c^2e]a_k+[-acd-a^2e]b_k+[2ace+c^2d+abd]C_k+[(ab-c^2)c]d_k+[a(ab-c^2)]e_k$

$$\partial x/\partial\hat{\theta}_k = \{f_k/r_k^2 - \Sigma\, 1/r_j^4 \cdot \partial r_j^2/\partial\hat{\theta}_k \cdot h_j\}/(ab-c^2)^2$$
$$\partial y/\partial\hat{\theta}_k = \{g_k/r_k^2 - \Sigma\, 1/r_j^4 \cdot \partial r_j^2/\partial\hat{\theta}_k \cdot i_j\}/(ab-c^2)^2$$

Let F_k, G_k, H_k, and I_k be f_k, g_k, h_k, and i_k evaluted at the true as in the Taylor Series expansion.

Plugging in for $\partial\hat{r}_i^2/\partial\hat{\theta}_k$ and evaluating at the true yields

$$\partial x/\partial\hat{\theta}_k = \{F_k/R_k^2 - \sum_{j=1}^{n} 2\{((X-x_j)\partial x/\partial\hat{\theta}_k+(Y-y_j)\partial y/\partial\hat{\theta}_k)H_j/R_j^4\}/(AB-C^2)^2$$
$$\partial y/\partial\hat{\theta}_k = \{G_k/R_k^2 - \sum_{j=1}^{n} 2\{((X-x_j)\partial x/\partial\hat{\theta}_k+(Y-y_j)\partial y/\partial\hat{\theta}_k)I_j/R_j^4\}/(AB-C^2)^2$$

where the partials shown are also evaluated at the true.

Let

$$q = [(AB-C^2)^2+2\sum_{j=1}^{n}\{(X-x_j)H_j/R_j^4\}][(AB-C^2)^2+2\sum_{j=1}^{n}\{(Y-y_j)I_j/R_j^4\}]$$
$$-[2\sum_{j=1}^{n}\{(Y-y_j)H_j/R_j^4\}][2\sum_{j=1}^{n}\{(X-x_j)I_j/R_j^4\}]$$

For partials evaluated at the true

$$\partial x/\partial\hat{\theta}_k = [F_k/R_k^2\{(AB-C^2)^2+\sum_{j=1}^{n}2\{((Y-y_j)I_j/R_j^4\}-G_k/R_k^2\{\sum_{j=1}^{n}2(Y-y_j)H_j/R_j^4\}]/q$$

$$\partial y/\partial\hat{\theta}_k = [-F_k/R_k^2\{\sum_{j=1}^{n}2\{((X-x_j)I_j/R_j^4\}+G_k/R_k^2\{(AB-C^2)^2+\sum_{j=1}^{n}2(X-x_j)H_j/R_j^4\}]/q$$

The answer must be invariant under translations (independent of where the origin is placed.) Fortunately calculations are a good deal easier if we choose $(X,Y)=(0,0)$. (Note we don't do that in the computerized examples however.) With this assumption

$D_k=E_k=0$ for all k. Hence $D=E=0$. Further implying $H_k=I_k=0$ for all k.

Hence $q=(AB-C^2)^4$ and

$$\partial x/\partial\hat{\theta}_k=(AB-C^2)^2F_k/(R_k^2q)=\{B(-2C_kx_k+(B_k-A_k)y_k)+C((B_k-A_k)x_k+2C_ky_k)\}/[R_k^2(AB-C^2)]$$

$$= \{B(-(B_k+A_k)y_k)+C((B_k+A_k)x_k\}/[R_k^2(AB-C^2)]$$
$$= \{-By_k+Cx_k\}/[R_k^2(AB-C^2)]$$

$$\partial y/\partial\hat{\theta}_k=(AB-C^2)^2G_k/(R_k^2q)=\{C(-2C_kx_k+(B_k-A_k)y_k)+A((B_k-A_k)x_k+2C_ky_k)\}/[R_k^2(AB-C^2)]$$

$$= \{-Cy_k+Ax_k\}/[R_k^2(AB-C^2)]$$

$$\{(Y-y_k)\partial x/\partial\hat{\theta}_j-(X-x_k)\partial y/\partial\hat{\theta}_j\}= -y_k\partial x/\partial\hat{\theta}_j+x_k\partial y/\partial\hat{\theta}_j$$
$$= \{By_k^2-2Cx_ky_k+Ax_k^2\}/[R_k^2(AB-C^2)]$$

$$\sum_{k=1}^{n}\{(Y-y_k)\partial x/\partial\hat{\theta}_k-(X-x_k)\partial y/\partial\hat{\theta}_k\}/R_k^2 = \{BA-2C^2+AB\}/(AB-C^2)=2$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\{(Y-y_k)\partial x/\partial\hat{\theta}_j-(X-x_k)\partial y/\partial\hat{\theta}_j\}^2/R_k^4$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\{y_k^2(\partial x/\partial\theta_j)^2-2x_ky_k(\partial x/\partial\theta_j)(\partial y/\partial\theta_j)+x_k^2(\partial y/\partial\theta_j)^2\}/R_k^4$$
$$= \sum_{j=1}^{n}\{A(\partial x/\partial\theta_j)^2-2C(\partial x/\partial\theta_j)(\partial y/\partial\theta_j)+B(\partial y/\partial\theta_j)^2\}$$
$$= \sum_{j=1}^{n}[A\{-By_j+Cx_j\}^2-2C\{-By_j+Cx_j\}\{-Cy_j+Ax_j\}+B\{-Cy_j+Ax_j\}^2]/[R_j^2(AB-C^2)]^2$$
$$=[A\{B^2A-2BCC+C^2B\}-2C\{BCA-BAC-CCC+CAB\}+B\{C^2A-2CAC+A^2B\}]/(AB-C^2)^2$$
$$=[AB\{AB-C^2\}-2C^2\{BA-C^2\}+AB\{-C^2+AB\}]/(AB-C^2)^2$$
$$=2$$

Thus

$$E(\sum_{k=1}^{n}(\theta_k(x,y)-\hat{\theta}_k)^2) \sim 2\sigma^2-2(2\sigma^2)+n\sigma^2 = (n-2)\sigma^2$$

(Note that partials with respect to θ_k are equal to partials with respect to ε_k.)

END
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